Modelling Extreme Mass Ratio Inspirals for LISA

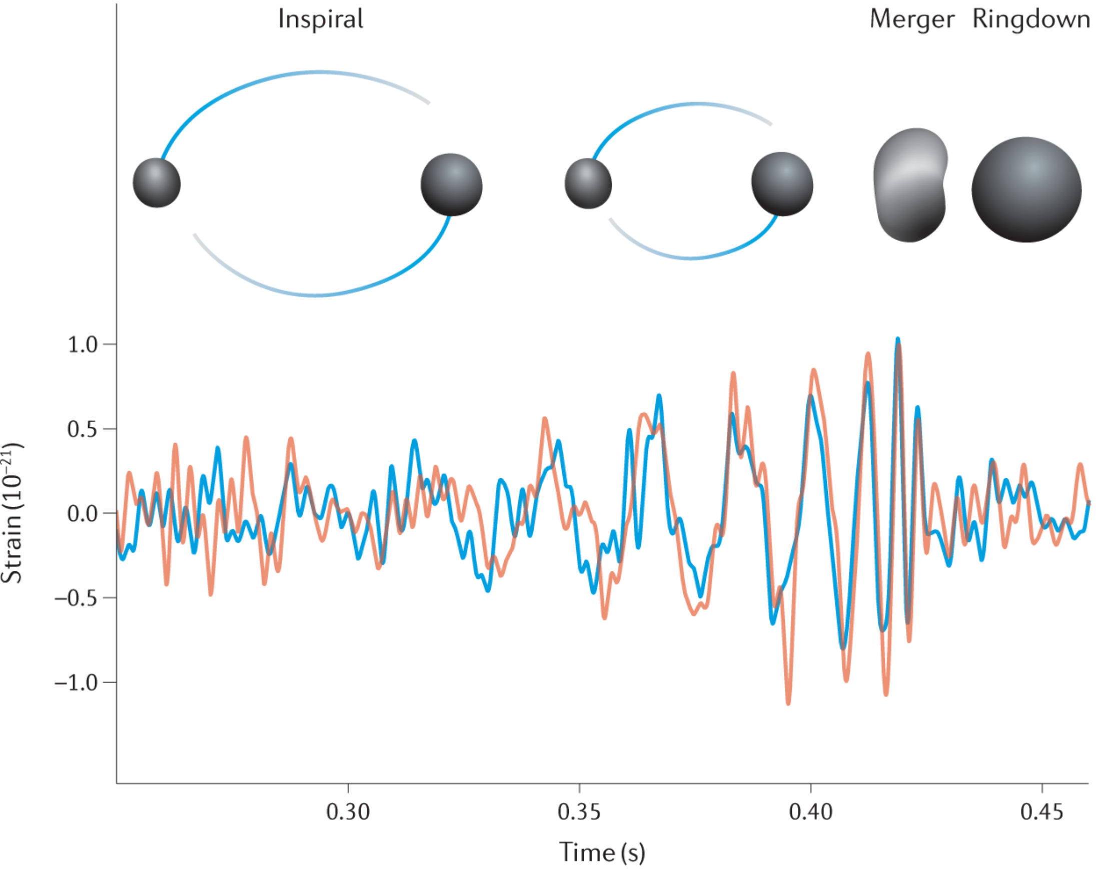
Overview, Challenges, and Roadmap

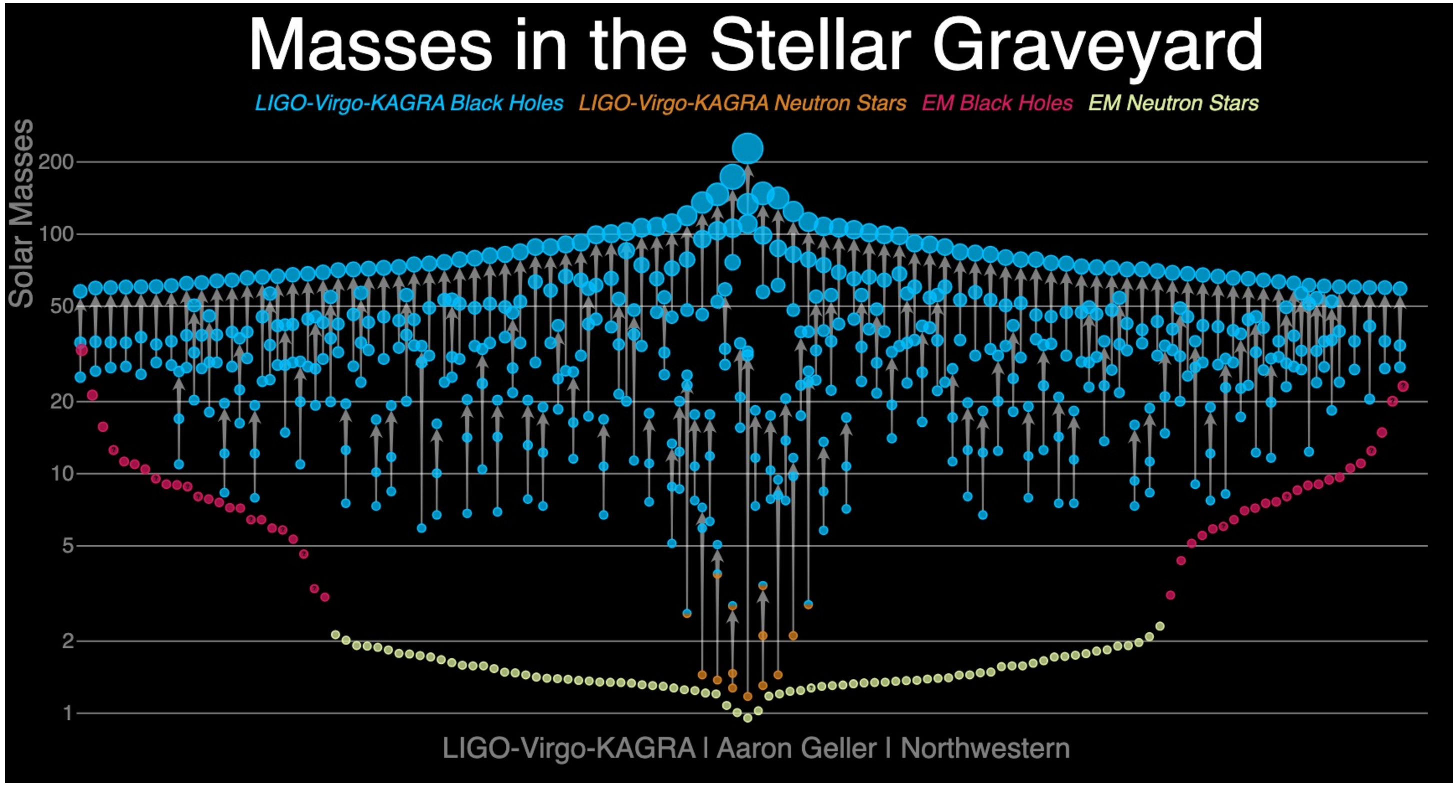
By Philip Lynch

About me

- 2014-18: **Undergrad in UCD**, Ireland
- 2018-22: **PhD in UCD** w/ Niels Warburton (LISA Consortium Spokesperson)
- 2022- 2027: Postdoc in MPI for Gravitational Physics (AEI), Potsdam
- 2022: Joined **FastEMRIWaveforms dev team**
- 2024: Member of LISA's DDPC, CU Wav
- **Co-Lead of the EMRI Subunit** w/ Adam Pound

GW Astronomy

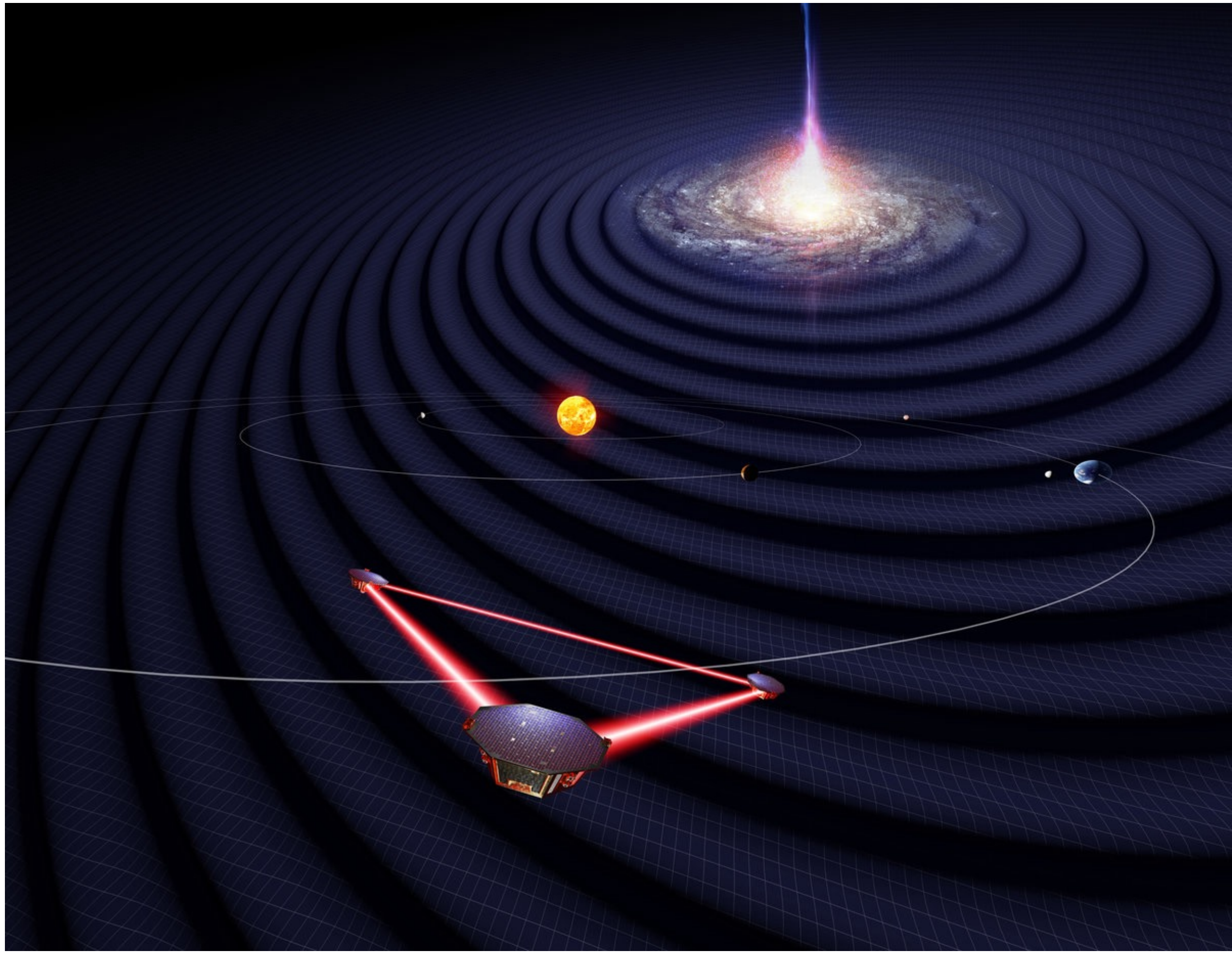




LVK Sensitivity: 10 – 10kHz

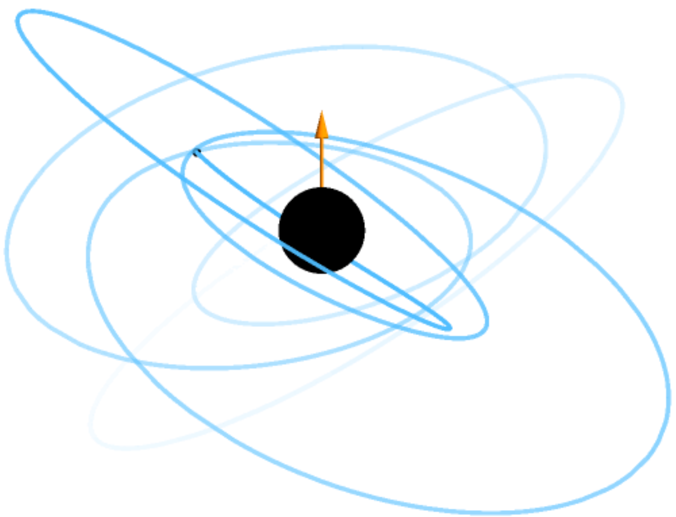
Space Based Detectors

- LISA, TianQin & Taiji
- 2024: LISA formally **adopted by ESA**
- Scheduled to luanch in **2035**
- Sensitive to **0.1 mHz to 1 Hz**
- See early inspiral of LVK sources
- **New GW Sources**: MBHBs, GBs, EMRIs & cosmological background etc.



Extreme Mass Ratio Inspirals (EMRIs)

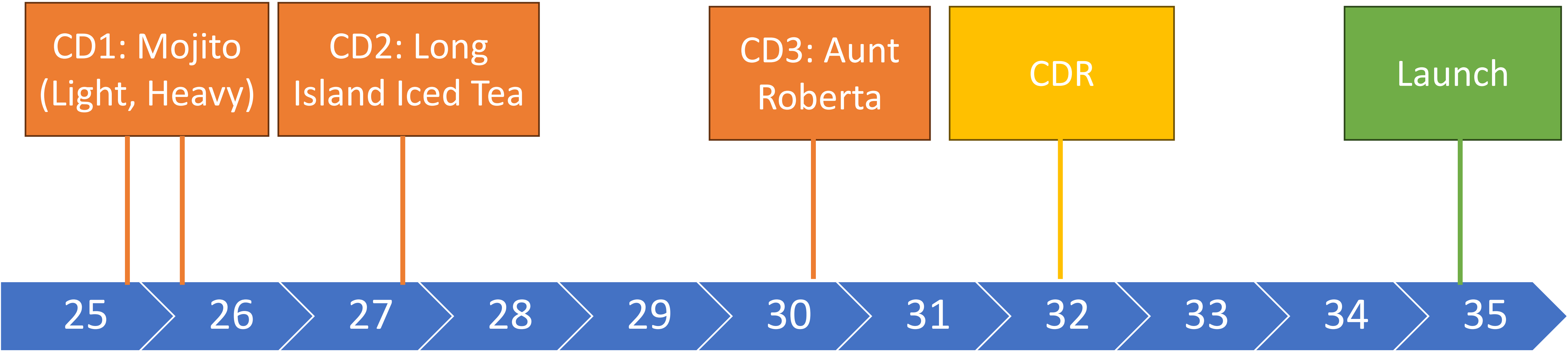
- **Massive black hole** (MBH) $\mathrm{m}_1 \sim 10^5\ M_\odot - 10^9 M_\odot$
- Compact object $\mathrm{m}_2 \sim M_\odot - 10^2\ M_\odot$
- **Small Mass Ratio**: $\epsilon = 10^{-4} - 10^{-7}$
- Loses energy and angular momentum to **GWs**
- Will stay in band for **years**!
- Expect ***eccentric***, ***inclined***, ***fast spinning*** primary $a \lesssim m_1$
- Expect **0 –1000s** during LISA
- Accurate MBH param est. and stringent tests of GR



EMRI Waveform Model Requirements

- **Extensive**
  - Cover full parameter space and include all necissary physics
  - Primary spin, eccentricity, inclination, transient resonances & sec. spin
- **Fast**
  - Needed to do Bayseian Param est. MCMC -> Millions of WF evaluations
  - Generate a full WF in ideall $\sim 10ms$ but at least $< 1s$
- **Accurate**
  - Error in the phase < 1 radian over a 4 year inspiral
  - Can get away with larger errors in the amplitudes
  - Most of the SNR in the inspiral, don’t need the merger ringdown
- **It has to be finished on time!**

LISA Pre-Mission Timeline



Relativistic Two Body Problem

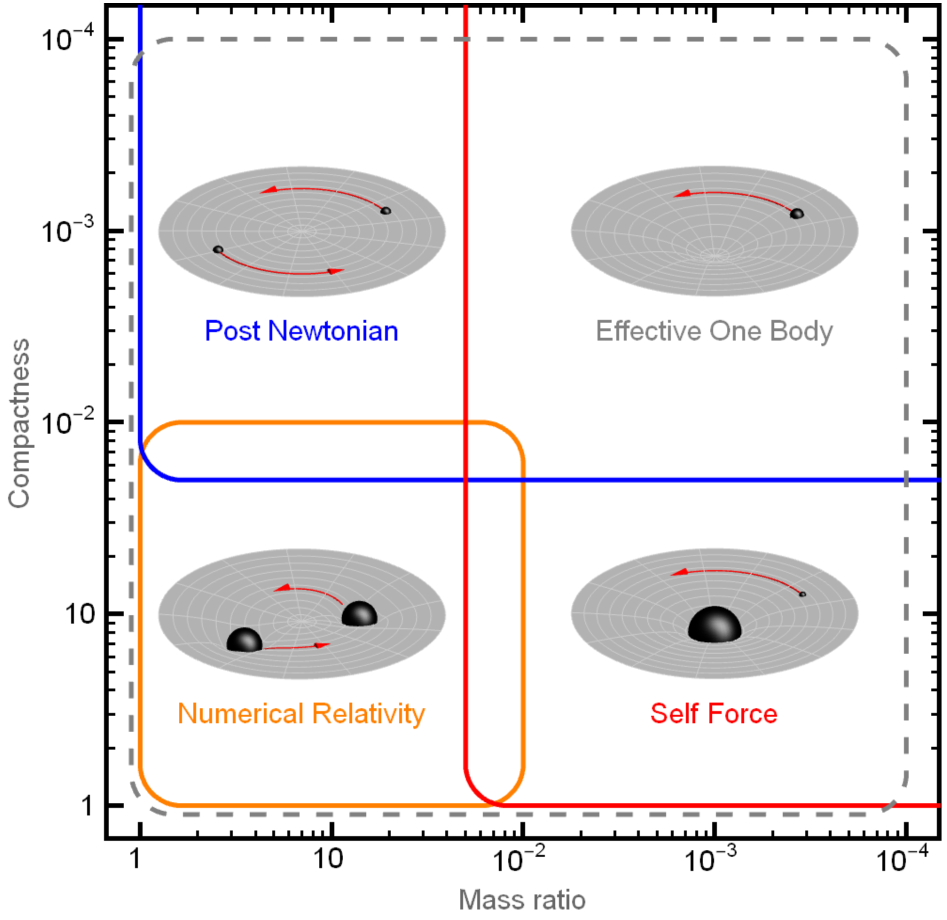
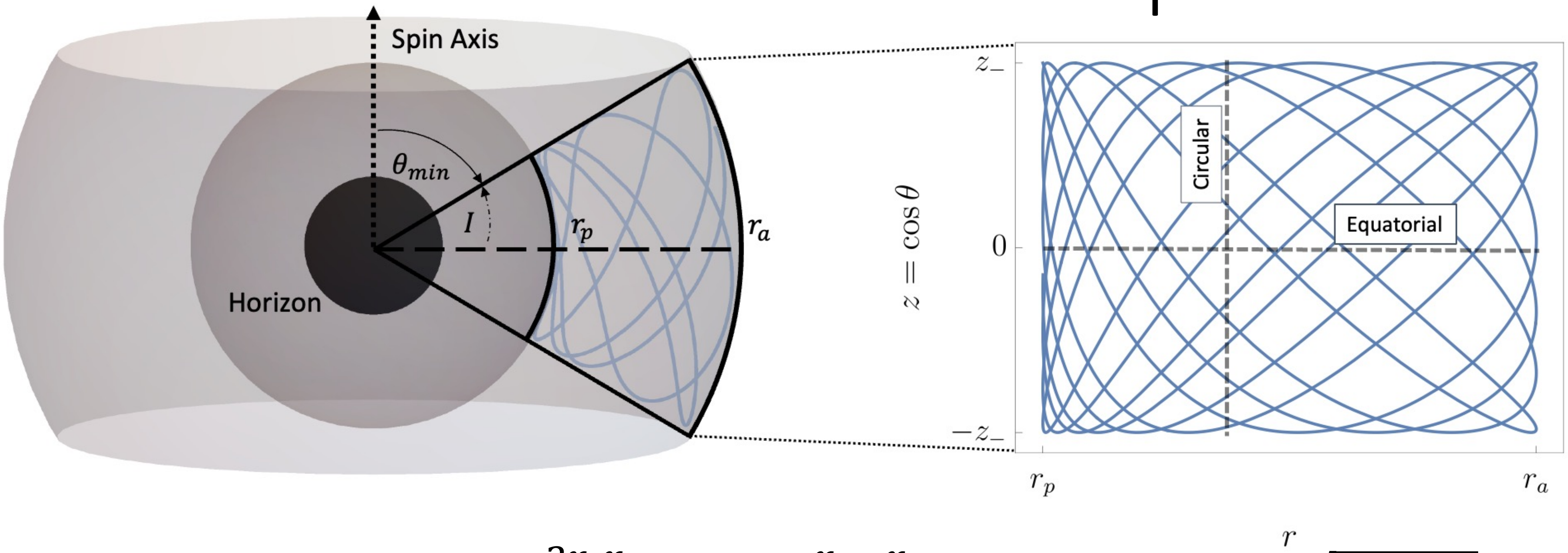


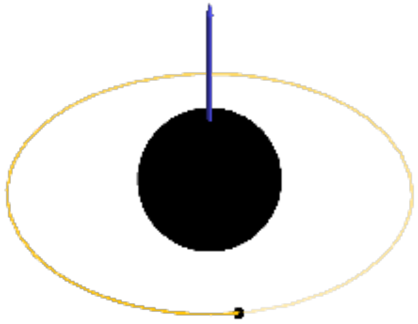
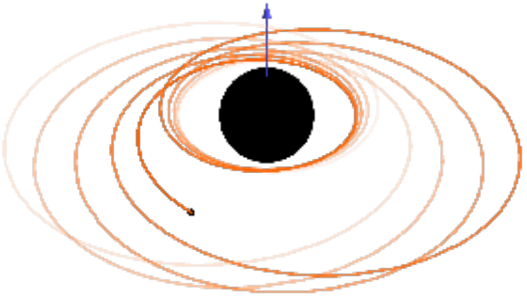
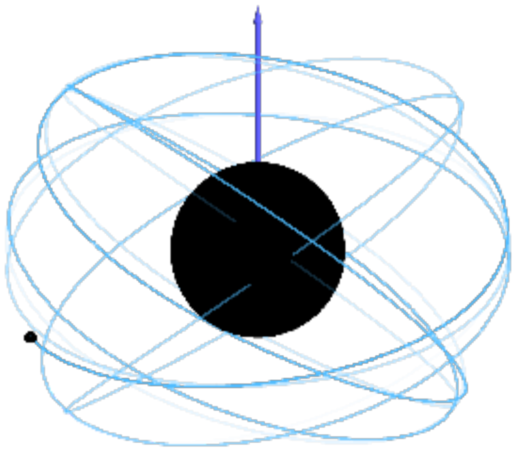
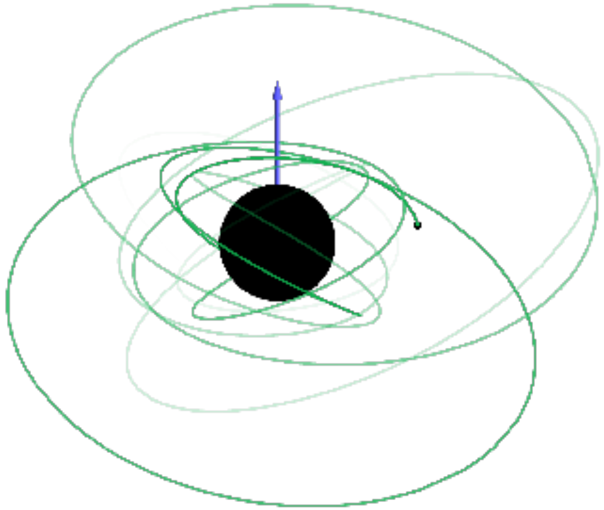
Image Credit: M. van de Meent

- NR: Solve EFEs numerically
- Post-Newtonian: Expand in $({}^{v}/_{c})^2$
- Post-Minkoskian: Expand in G
- EOB: Use information from the above
- Self-force: Expand in $\epsilon = m_2/m_1 \in [0,1]$
- Symmetric mass ratio $\nu = \frac{m_2 m_1}{(m_1+m_2)^2} \in [0, ¼]$

Geodesic Motion in Kerr Spacetime



- **Orbital Elements** P_j: $p = \frac{2r_1 r_2}{M(r_1+r_2)}$, $e = \frac{r_1 - r_2}{r_1+r_2}$, $x = \cos\theta_{inc} = \sqrt{1 - z_-^2}$
- **Orbital phases** Φ_A : $\frac{d\Phi_A}{dt} = \Omega_A^{(0)}(a, p, e, x)$
- Analytic solutions in terms of Jacobi Elliptic Integrals [Fujita & Hikida 09]
- Efficient conversions between Phases and Coordinates [PL & Burke 24]

	Quasi-Circular $e = 0$	Eccentric $0 < e < 1$
Equatorial $x = \pm 1$		
Inclined $-1 < x < 1$	 (a.k.a Spherical)	 (a.k.a Generic)

Gravitational Self-Force

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = \epsilon a^\alpha_{(1)} + \epsilon^2 a^\alpha_{(2)} + \mathcal{O}(\epsilon^3)$$

Two-timescale analysis [Hinder & Flannigan 08]:

$$\varphi = \epsilon^{-1}\varphi^{(0\mathrm{PA})} + \epsilon^{-1/2}\varphi^{(\mathrm{res})} + \varphi^{(1\mathrm{PA})} + \mathcal{O}(\epsilon)$$

- Adiabatic (0PA): $< a^{(1)}_{Diss} >$ or $\mathcal{E}$ & $\mathcal{L}$ Fluxes
- Post-Adiabatic (1PA): $a^{(1)}$ & $< a^{(2)}_{Diss} >$ & Sec. Spin
- Orbital Resonances: Generic Kerr only

Adiabatic Inspirals (0PA)

Teukolsky Equation

$$\mathcal{O}\,\psi_4 = 4\,\pi\Sigma\,\mathcal{T}$$

$$\psi_4 = \frac{1}{2}\frac{d^2}{dt^2}\left(h_+ - i\,h_\times\right) \text{ as } r \;\to \infty$$

Solve in Frquency Domain with a geodesic source

Waveform Amplitudes

Multi-voice decomposition [Hughes +20]

$$h = \frac{m_2}{d_L}\sum_{lmkn} \mathcal{A}_{lmnk}(\vec{P})\,{}_{-2}Y_{lm}\,(\Theta,\Phi)\exp(-i\,\Phi_{mkn})$$

where $\Phi_{mkn} = m\,\Phi_\phi + k\,\Phi_\theta + n\,\Phi_r$

High dimensionality, lax accuracy requirement

ML techiques could be useful here

Asymptotic Fluxes

$< \dot{\mathcal{J}} >^{\infty}$ & $< \dot{\mathcal{J}} >^{\mathcal{H}}$ where $\mathcal{J} = (\mathcal{E}, \mathcal{L}_z, \mathcal{Q})$

Flux Balance Laws + Geodesic Relations

$\rightarrow \dot{P}_{\boldsymbol{j}}$

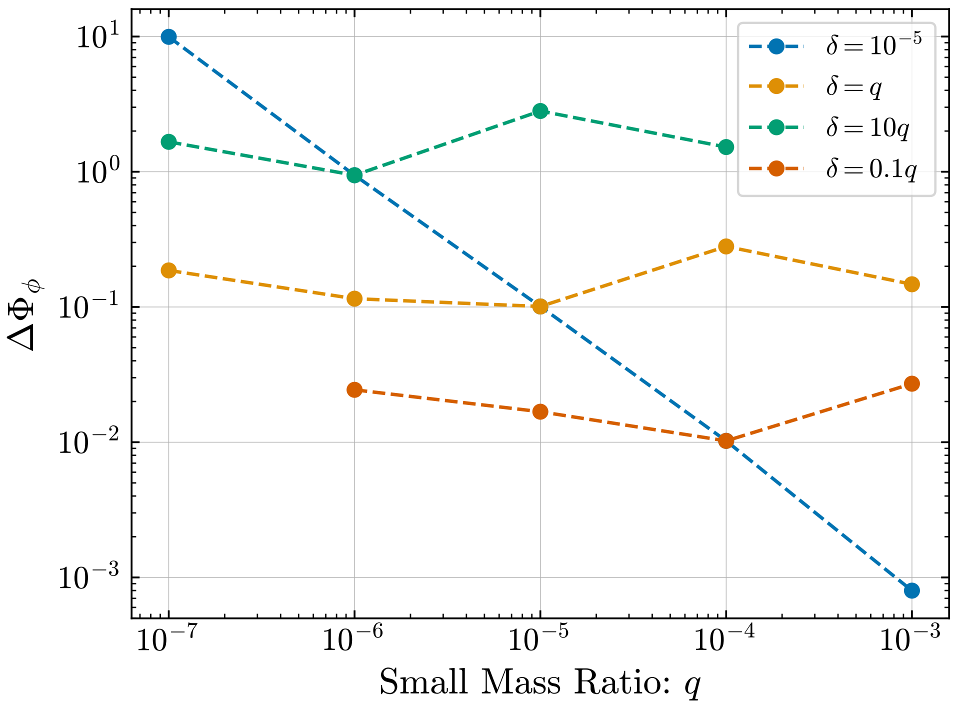
Low dimensionality, high accuracy requirement

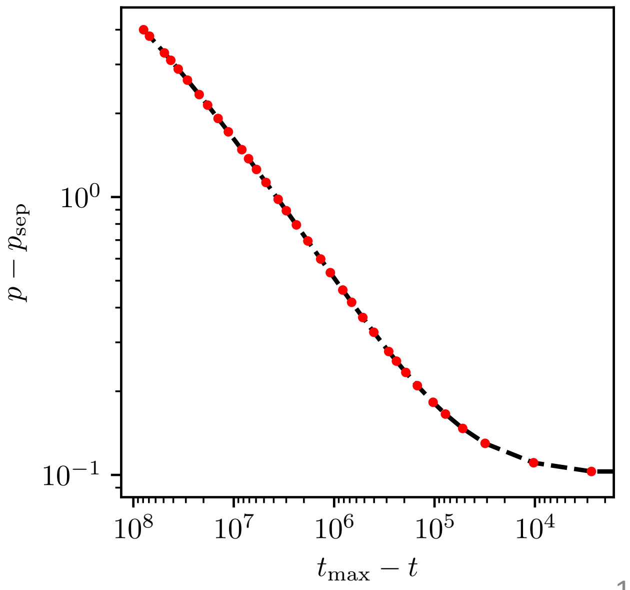
ML will struggle

Interpolation of the fluxes

- Solve numerically at a single (a,p,e,x)
- **Offline step** (expensive, once):
  - Tile the parameter space and interpolate fluxes + mode amplitudes
  - **Flux rel. error** $\delta \leq \epsilon \leq 10^{-6}$ [Khavali, PL+ 2025]
  - Amplitudes can be much less accurate
- **Online step**: Numerically Solve ODEs

$$\dot{P}_j = \nu\, F_j^{(1)}(\vec{P}) + \mathcal{O}(\nu^2)$$
$$\dot{\Phi}_A = \Omega_A^{(0)}(\vec{P}) + \mathcal{O}(\nu)$$





$$(\vec{P}(t), \vec{\Phi}(t))$$

Sub $\vec{P}(t)$ into $\mathcal{A}_{lmnk}(t)$

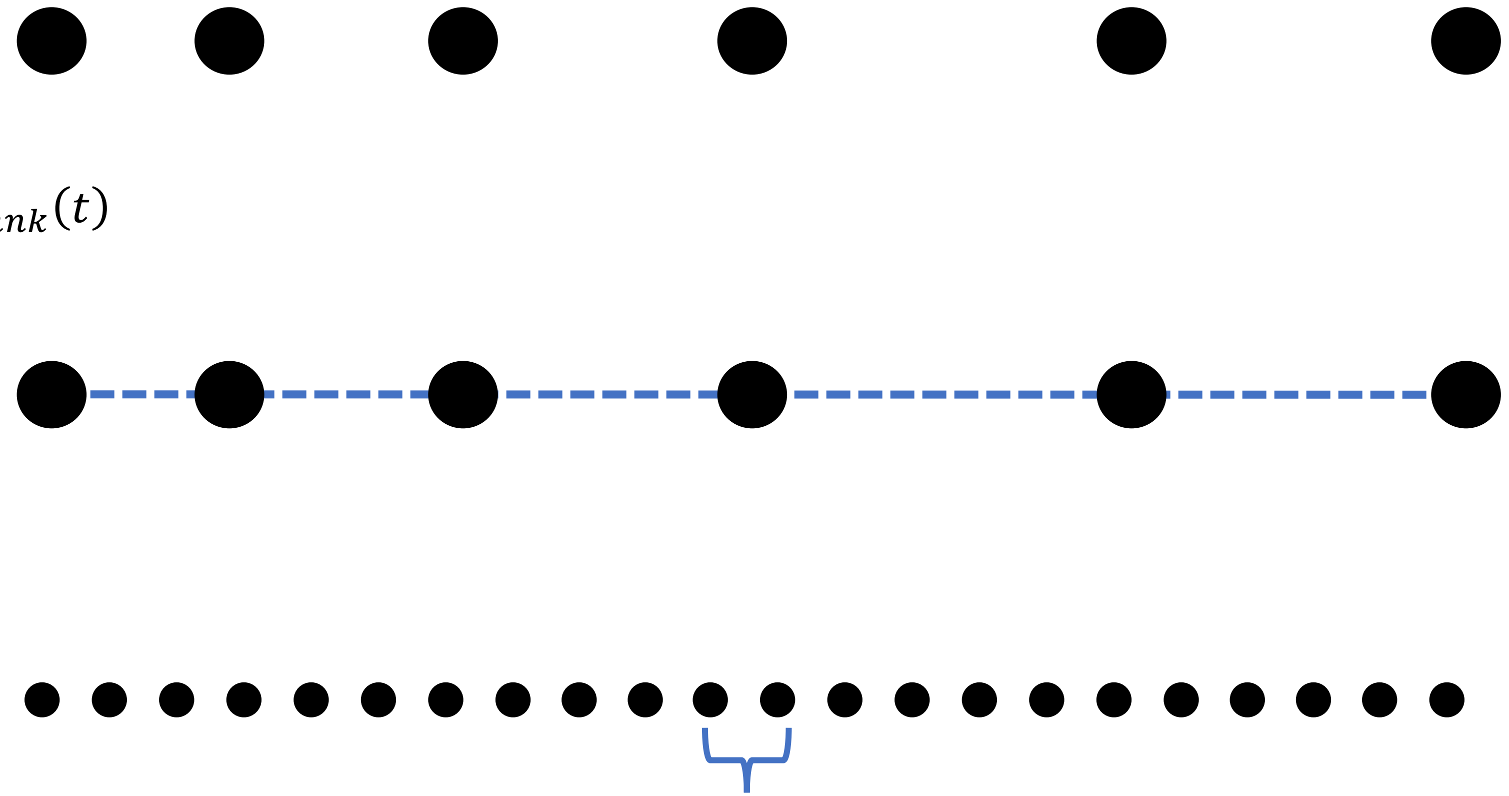
$$(\mathcal{A}_{lmnk}(t), \vec{\Phi}(t))$$

Interpolate

$$h = \frac{m_2}{d_L} \sum_{lmkn} \mathcal{A}_{lmnk\,-2}Y_{lm} e^{-i\,\Phi_{mkn}}$$

Summation

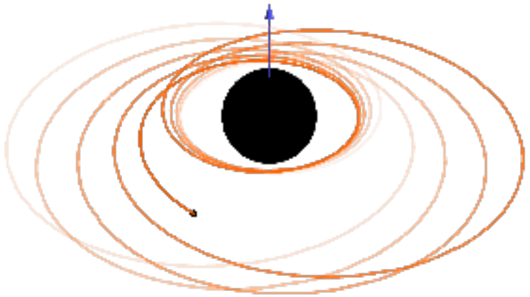
$$h\,(\Delta t)$$

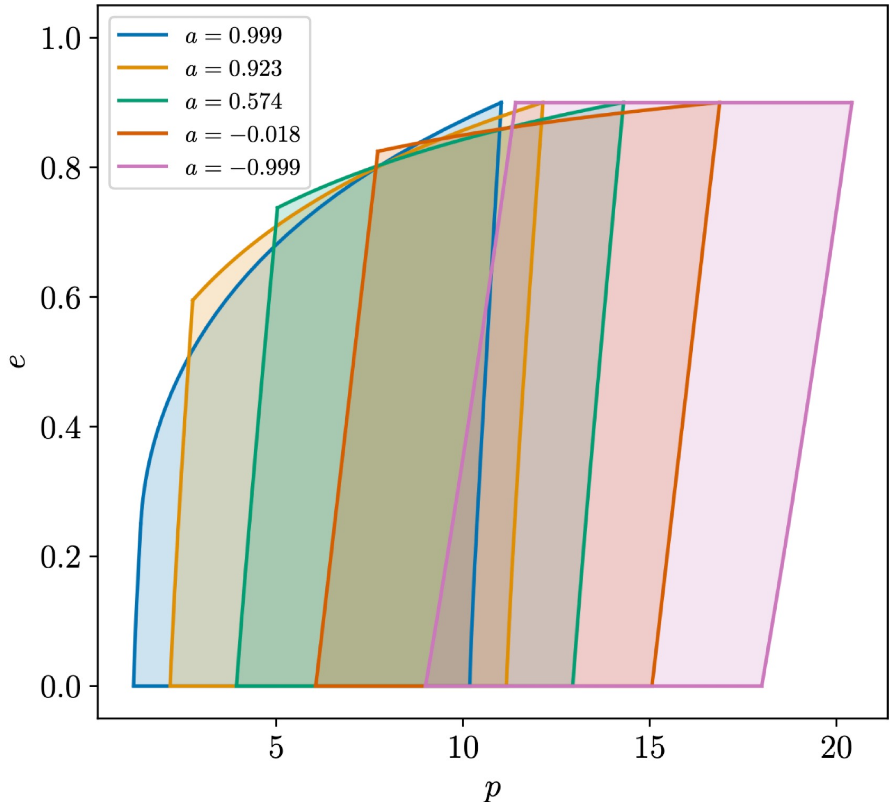


$$\Delta t \leq \frac{1}{2\, f_{max}}$$

FastEMRIWaveforms (FEW)

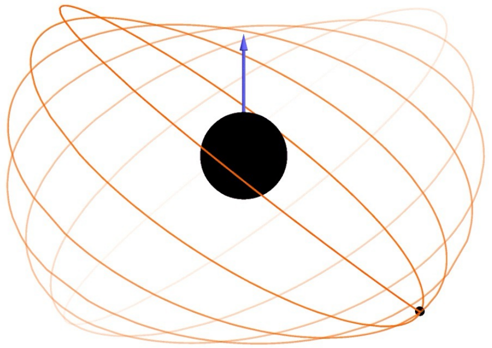
- Leverages **GPUs** for Parallel calculations
- Can now do **eccentric** inspirals in **Kerr** [Chapman-Bird +(PL) 25]
- $a \in [-0.999, 0.999], e \in [0, 0.9^*]$
- **Flux points**:
  - Inner (65 x 129 x 65)
  - Outer (33 x 65 x 33)
  - Total: 615,810 points costing 605,000 CPU hours
  - Interpolated with tri-cubic splines
- **Mode Amplitudes**
  - $\ell \in [2,10], m \in [0, \ell], n \in [-55,55]$
  - Total ~ 7000
  - Bi-cubic+linear spline
- **Mode Selector** only keeps necessary modes
- Speed: $\mathcal{O}(100ms)$



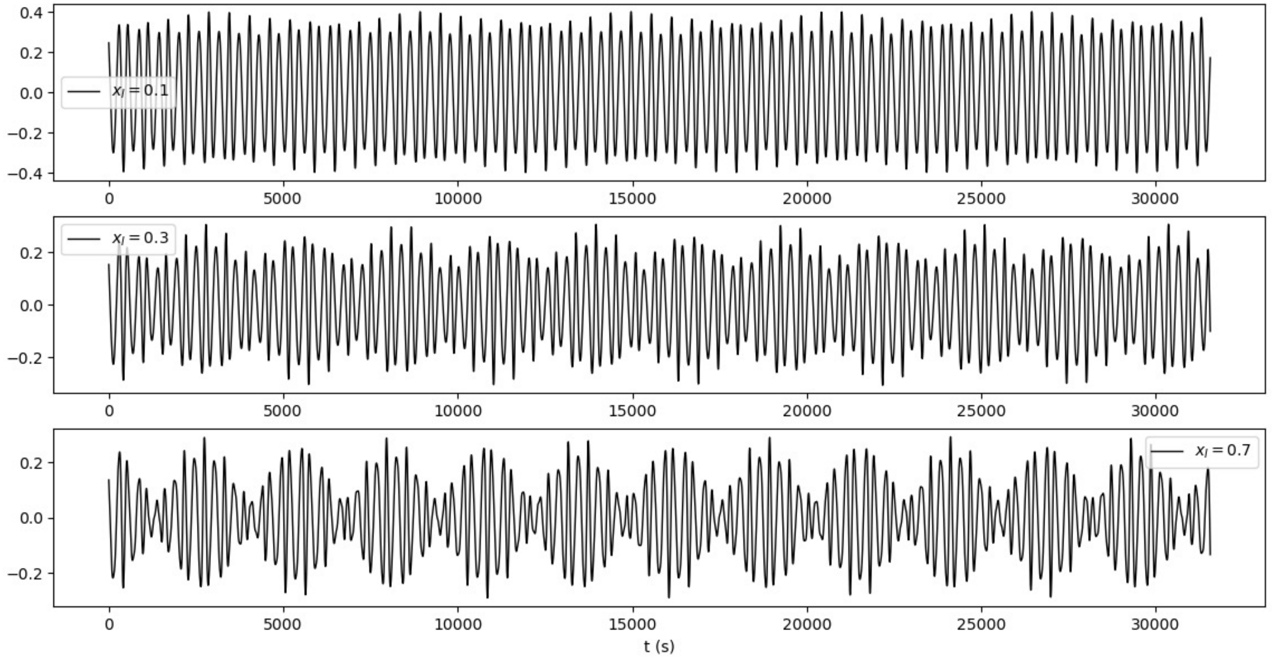


Spherical Orbits

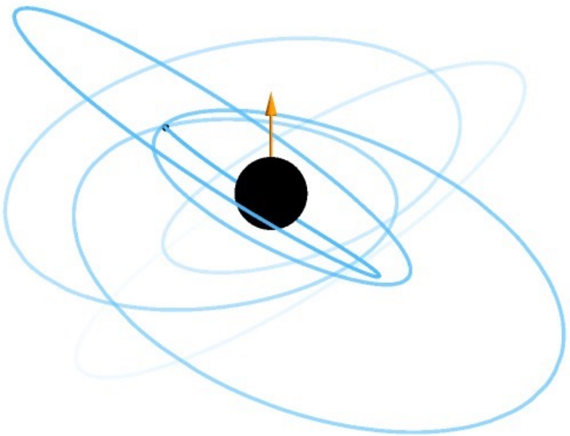
- FEW can now handle k modes
- Comparisons with high-order PN fluxes
- Cross checking interpolants with independent grids
- Checking spin precession conventions from comparable mass binaries (Leo Stein, Josh Mathews)
- Finish by Q1 2026



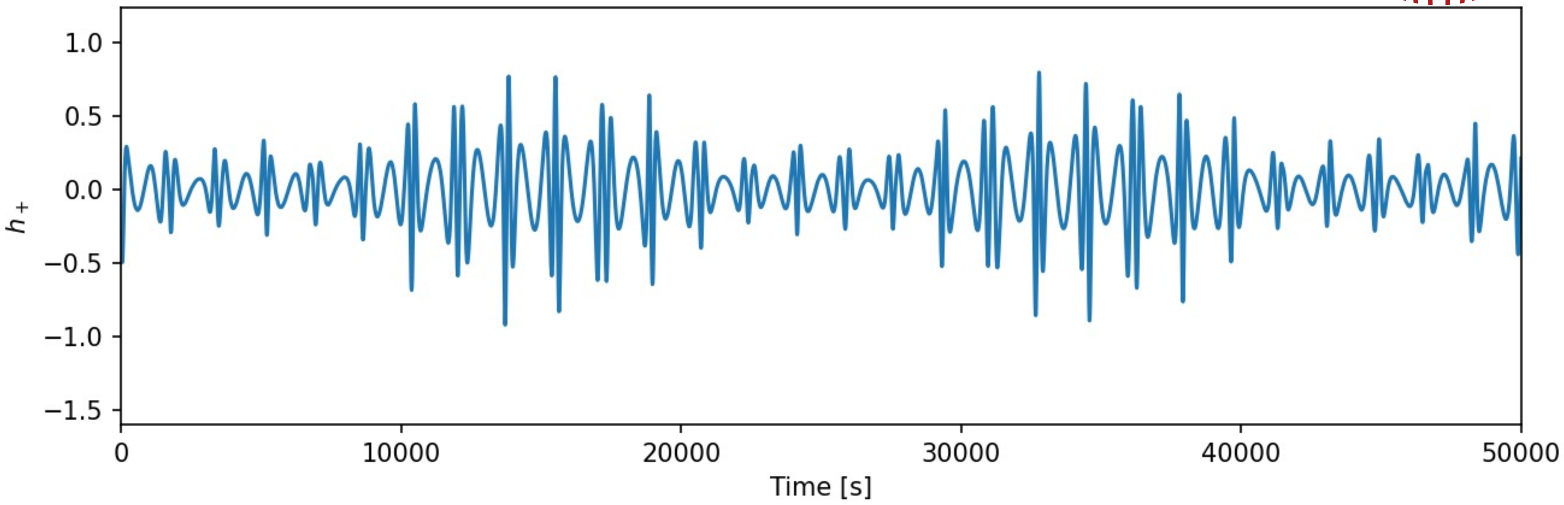
Varying initial inclination x_I (with a=0.5)

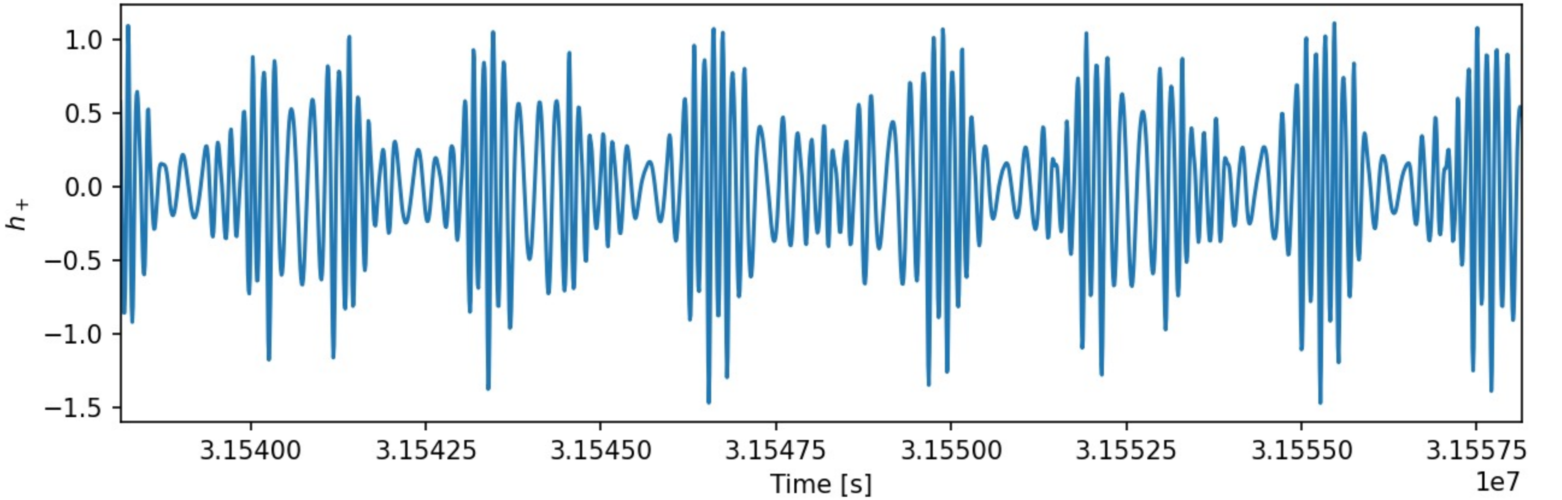


Generic Orbits

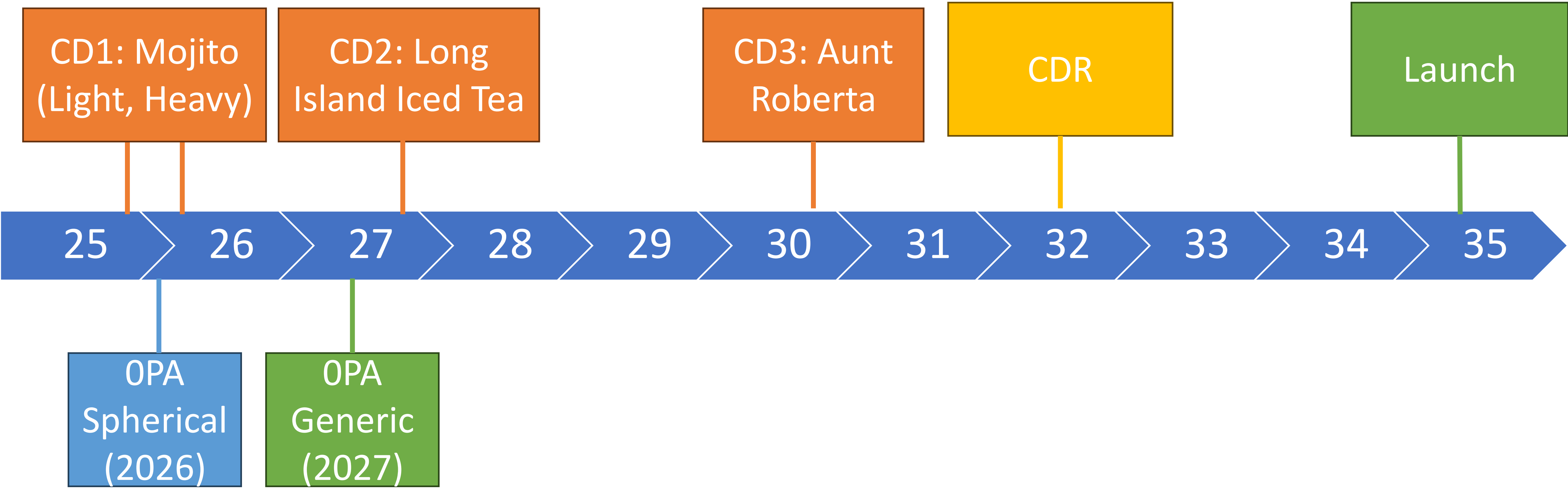


- Interpolated a = 0.7
- 4D spline interpolation now developed (Zach Nasipak)
- 4D Chebyshev interpolation developed (Philp Lynch)
- Still need to calculate the grid
- Aim for Q2 2027



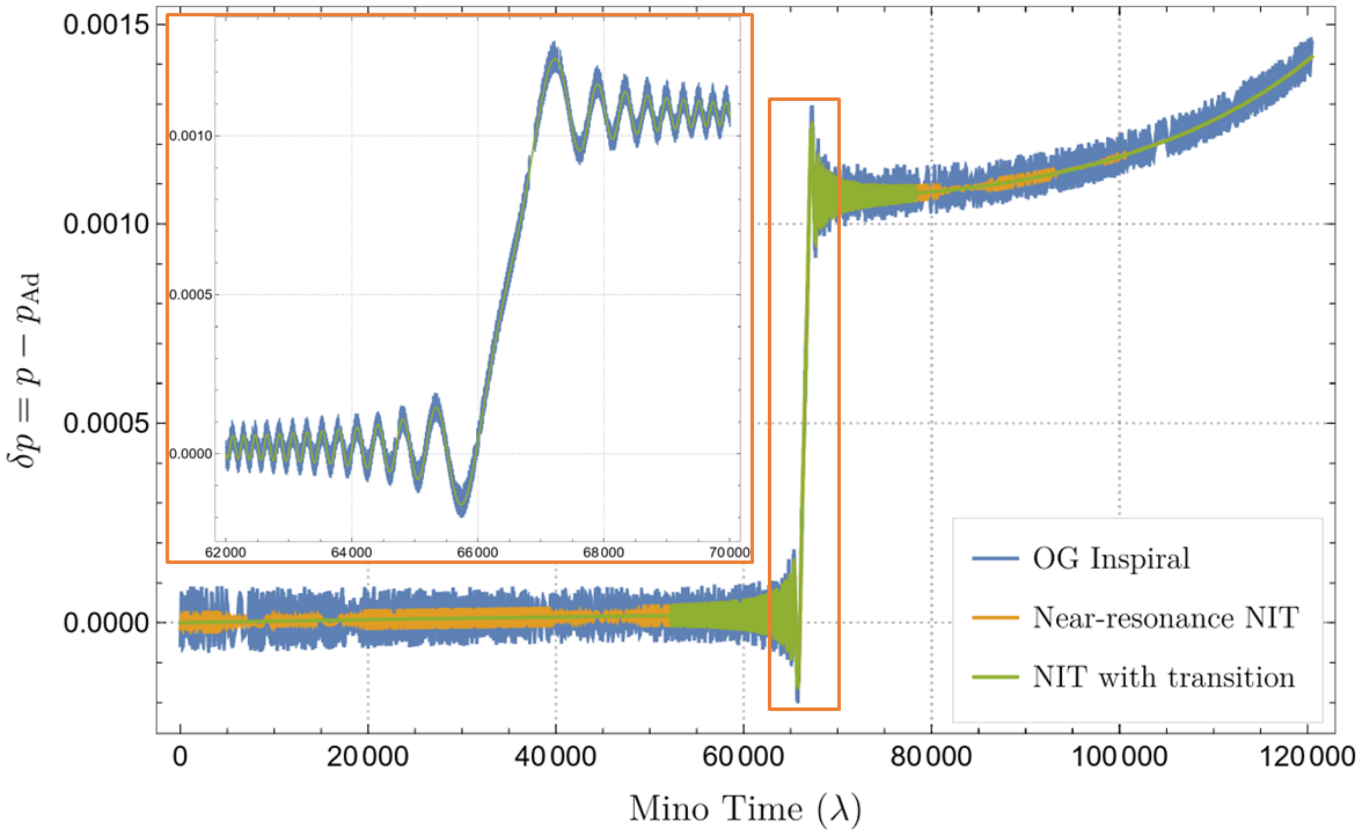


The FEWture Timeline



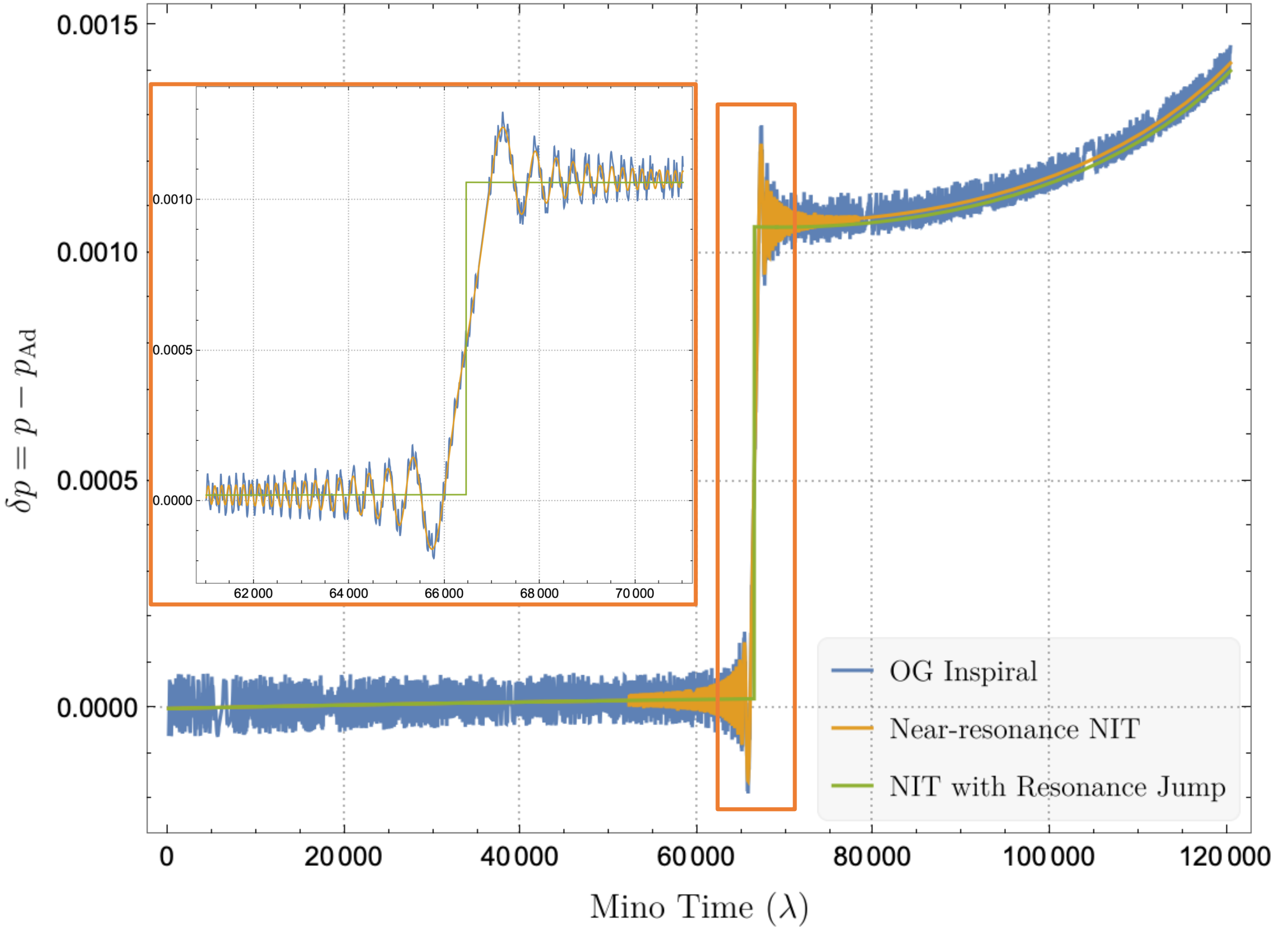
Transient Resonances (½ PA)

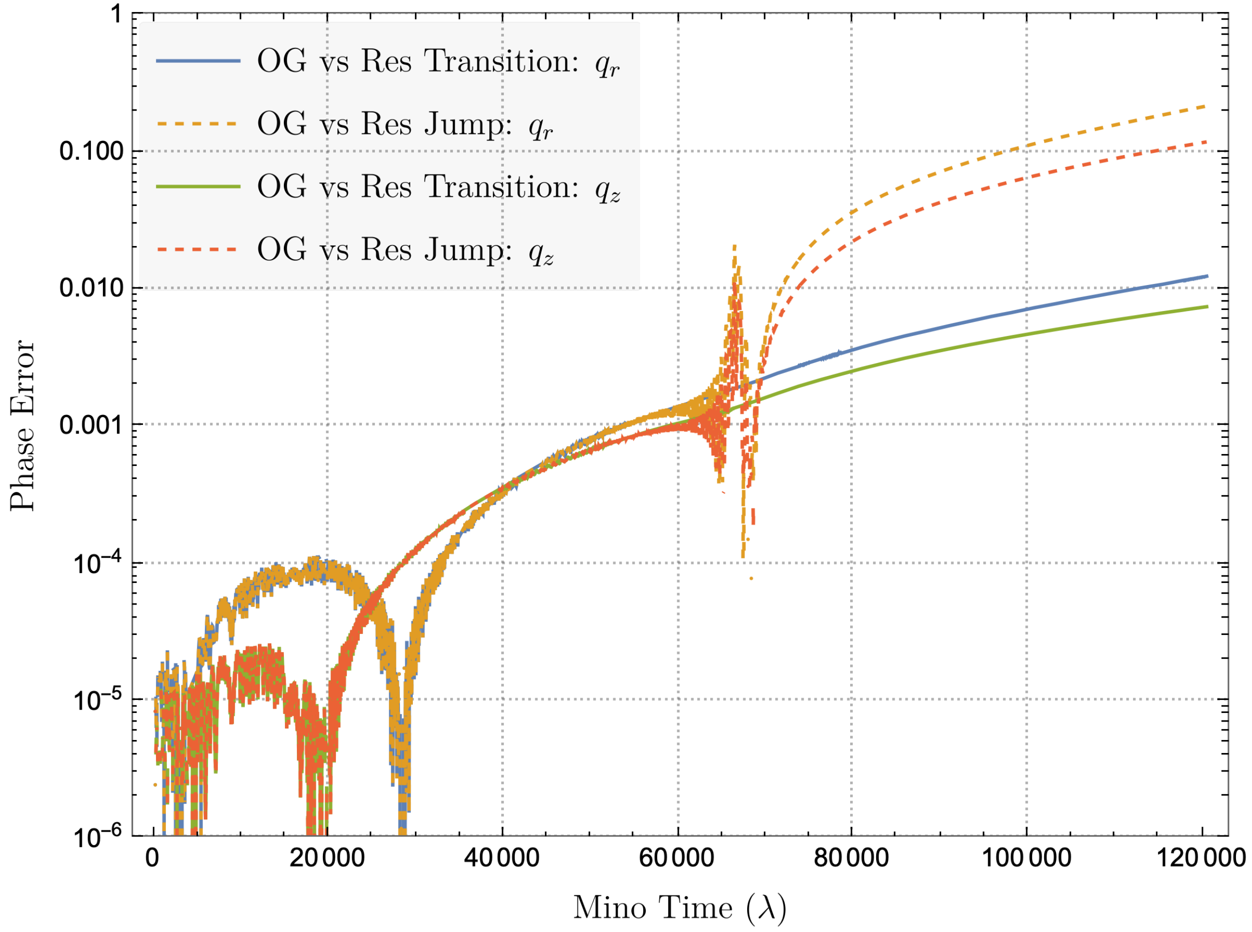
- Occur when $k\,\Omega_\theta - n\,\Omega_r = 0$ for $n, k \in \mathbb{N}$
- Cause "jumps" in $\vec{P} \propto \epsilon^{1/2}$
- Results in phase error $\propto \epsilon^{-1/2}$
- Work ongoing to model resonant jump in analytic Kludge model in FEW



Are Resonace Jumps Accurate enough for 1PA?

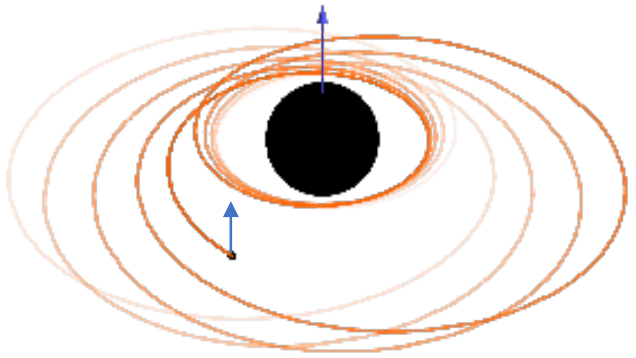
- Leading order Jump ΔP_j^{res} [Flannagan & Hinderer 12]
- Accurate Resonance treatment [Lynch+24]
- Vojtech and I are looking to find subleading piece needed for accurate 1PA inspirals





Spinning Secondary (1PA)

- For EMRIs, only need **linear in sec. spin**
- **Flux Balance Law** valid for spinning bodies in **generic orbits** [Grant 24]
- **Analytic Solutions** for Spinning Test Bodies [Skoupy & Witzany 25]
- Solve Teukolsky for a spinning test body and linearize in sec. spin
- Curently adding linear in spin corrections to **eccentric orbits** into FEW



$$\dot{P}_j = \nu\, F_j^{(1)}(\vec{P}) + \nu^2 s_\parallel F_j^{(s)}(\vec{P})$$
$$\dot{\Phi}_A = \Omega_A^{(0)}(\vec{P}) + \nu s_\parallel \Omega_A^{(s)}(\vec{P})$$

Second Order GSF (1PA)

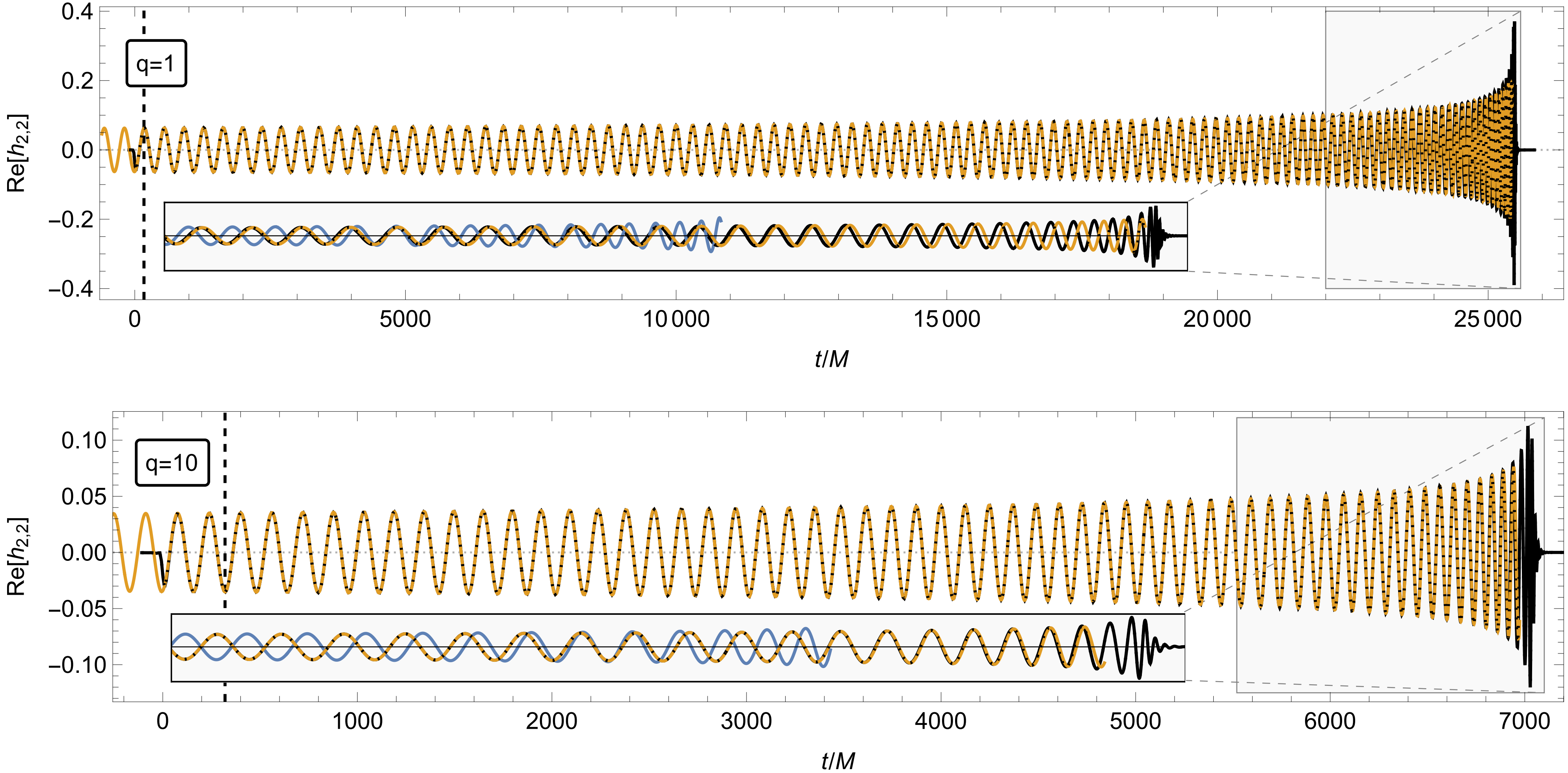
- $g^{exact}_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \epsilon\, h^{(1)}_{\alpha\beta} + \epsilon^2\, h^{(2)}_{\alpha\beta}$
- Split into Regular and Singular pieces
- Sub into EFEs: $G_{\alpha\beta}[g] = 8\,\pi T_{\alpha\beta}$
- ϵ^0: $G_{\alpha\beta}\left[g^{(0)}_{\alpha\beta}\right] = 0$
- ϵ^1: $G^{(1)}_{\alpha\beta}\left[h^{(1)R}_{\alpha\beta}\right] = T^{(1)}_{\alpha\beta}[z] - G^{(1)}_{\alpha\beta}\left[h^{(1)S}_{\alpha\beta}\right]$
- ϵ^2: $G^{(1)}_{\alpha\beta}\left[h^{(2)R}_{\alpha\beta}\right] = T^{(2)}_{\alpha\beta}[z] - G^{(2)}_{\alpha\beta}\left[h^{(1)}_{\alpha\beta}, h^{(1)}_{\alpha\beta}\right] - G^{(1)}_{\alpha\beta}\left[h^{(2)S}_{\alpha\beta}\right] - \partial_{\tilde{t}}\, h^{(1)}_{\alpha\beta}$

This is very hard – Barry Wardell (2025)

- Multiscale Self-Force Collaboration

$$\dot{P}_j = \nu\, F^{(1)}_j(\vec{P}) + \nu^2 F^{(2)}_j(\vec{P})$$
$$\dot{\Phi}_A = \Omega^{(0)}_A(\vec{P}) + \nu\, \Omega^{(1)}_A(\vec{P})$$

Quasi-circular non-spinning [Warburton & Pound & Wardell + 19,20,21]



Quasi-Circular

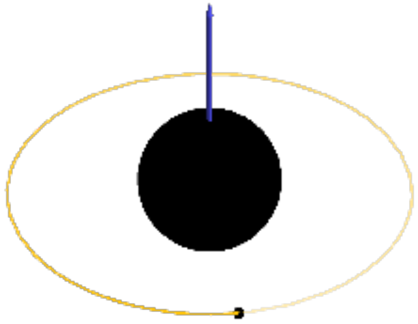
$e = 0$

Eccentric

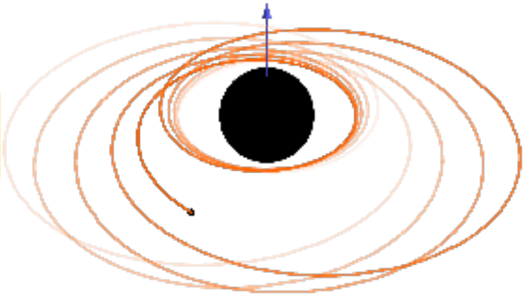
$0 < e < 1$

Equatorial

$x = \pm 1$



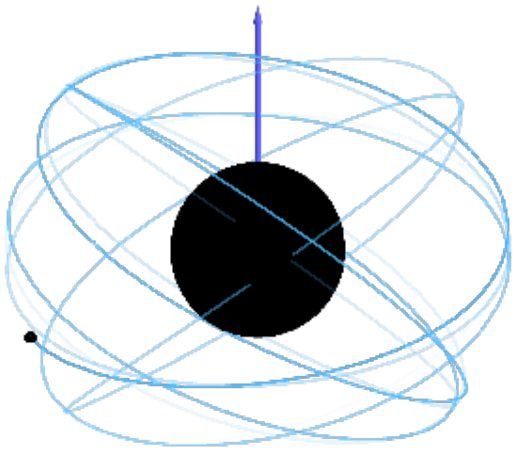
1PA: Q2 2027



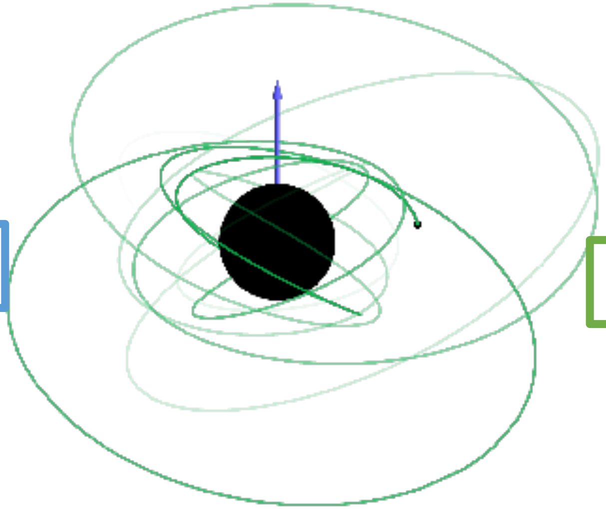
1PA ~ 2029 ?

Inclined

$-1 < x < 1$



1PA: Q2 2028

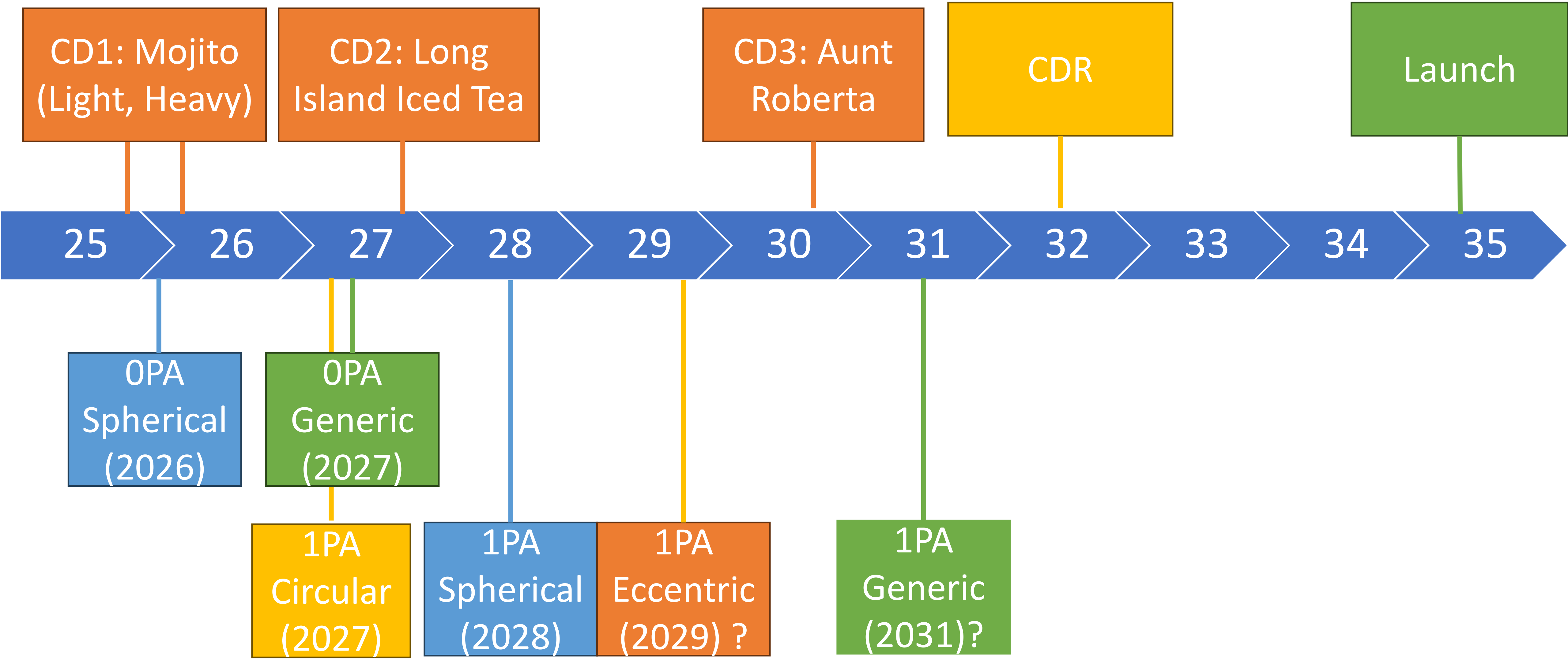


1PA ~ 2030??

(a.k.a Spherical)

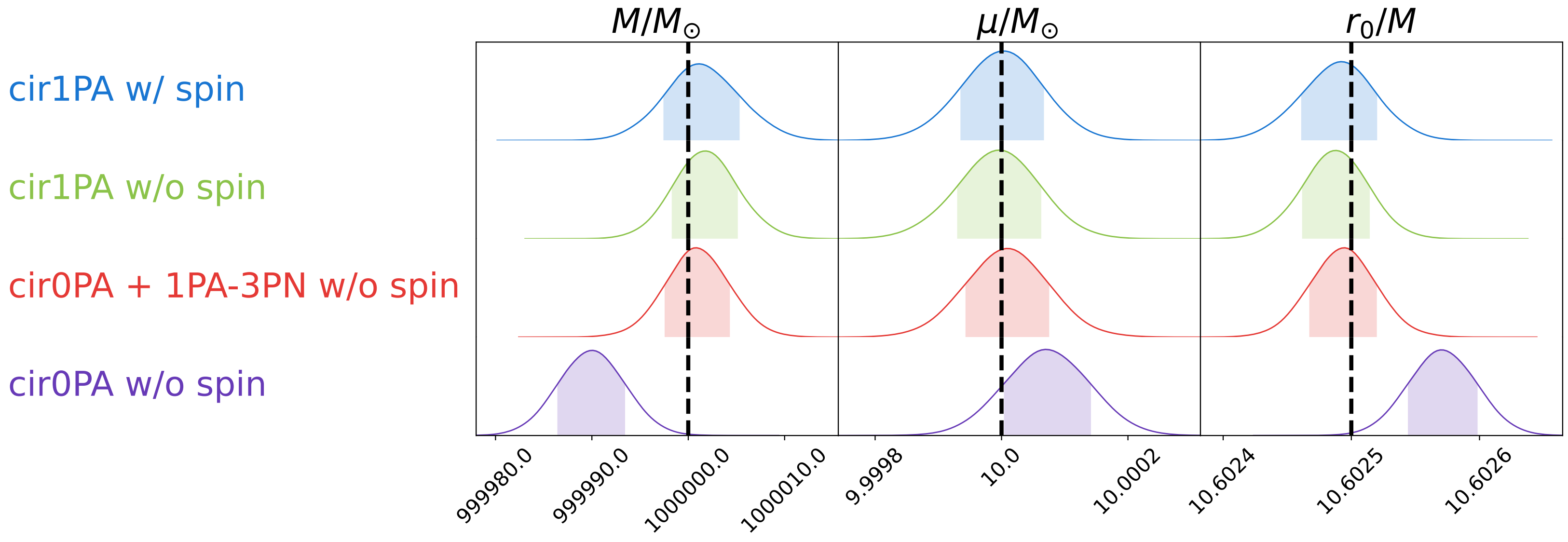
(a.k.a Generic)

The FEWture Timeline



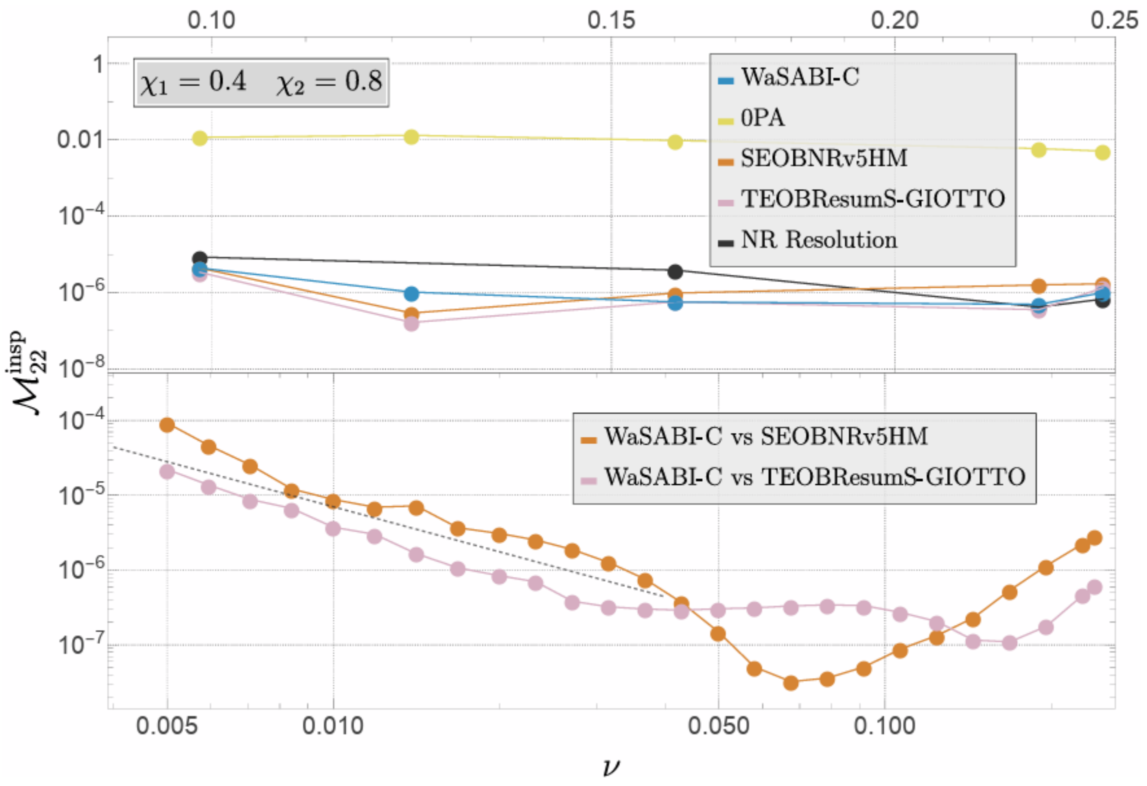
1PA Quasi-cirular Schwarzschild in FEW [Burke+ (PL) 24]

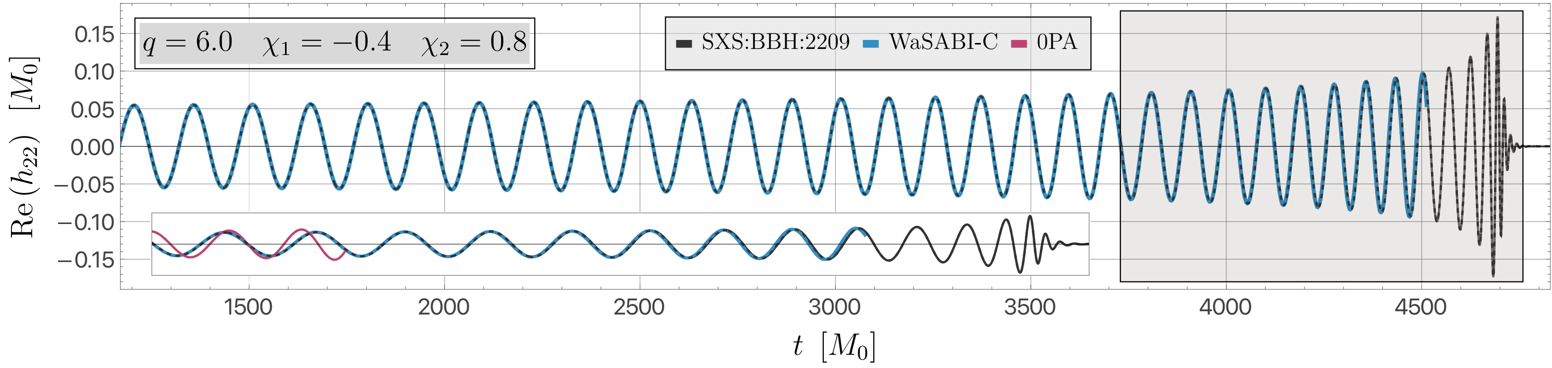
$$\dot{P}_j = \nu\, F_j^{(1)}(\vec{P}) + \nu^2 F_j^{(2)}(\vec{P}) + \nu^2 s_\parallel F_j^{(s)}(\vec{P})$$
$$\dot{\Phi}_A = \Omega_A^{(0)}(\vec{P}) + \nu\, \Omega_A^{(1)}(\vec{P}) + \nu s_\parallel \Omega_A^{(s)}(\vec{P})$$



Hybrid waveforms w/ PN

- Failsafe if we do not get complete 1PA
- First published model:
  - Slow spinning primary
  - Aligned Secondary Spin
  - 1PA and higher terms extracted from PN
- In EMRI limit: about 0.1 rad dephasing relative to complete 1PA model in nonspinning case
- Mathematica implementation: WaSABI





Summary

Adiabatic 0PA:

- FEW: $\mathcal{O}(100ms)$
- On track to cover spherical & generic by 2027
- Can easily add 1PA results

Resonace ½ PA:

- Implimentation in Kludge
  - May need to go past leading order jumps

Spinning Secondary 1 PA:

- Theory is done
- Starting on the practical
- Can be added modularly

2nd Order SF (1 PA):

- Slow but steady progress
- Spherical before eccentric
  - May need PN hybrids